Robust Reachable Set: Accounting for Uncertainties in Linear Dynamical Systems

EMSOFT 2019

Department of Computer Science

THE UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

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Motivation

- One of the most used techniques for safety verification.

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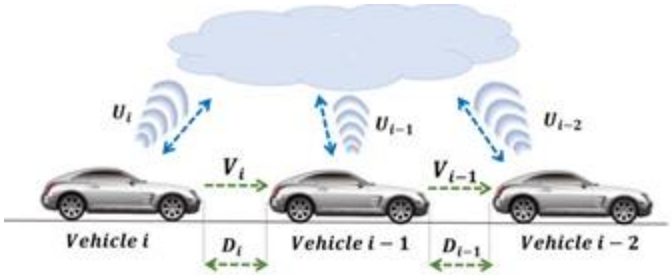

Modelling depends on constants like acceleration due to gravity, weight of the components *etc*.

- In reality the underlying dynamics have uncertainties like parameter variations or modelling uncertainties.

Motivation – Example of Anesthesia Model

A safety critical event that occurs before (almost) all surgeries.

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Determining its *safety*:

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- Understanding of how it is metabolized

Motivation – Example of Anesthesia Model

A safety critical event that occurs before (almost) all surgeries.

Determining its *safety*:

- Understanding of how it is metabolized
- Its affect on the depth of *hypnosis*

Motivation – Example of Anesthesia Model

Obtain a dynamical model

$$\begin{bmatrix} \dot{c}_p \\ \dot{c}_1 \\ \dot{c}_2 \\ \dot{c}_e \\ \dot{u} \end{bmatrix} = \begin{pmatrix} -(k_{10}+k_{12}+k_{13}) & k_{12} & k_{13} & 0 & 1/V_1 \\ k_{21} & -k_{21} & 0 & 0 & 0 \\ k_{31} & 0 & -k_{31} & 0 & 0 \\ k_d & 0 & 0 & -k_d & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} c_p \\ c_1 \\ c_2 \\ c_e \\ u \end{bmatrix}$$

Motivation – Example of Anesthesia Model

Perform analysis on this model

Safe

Unsafe

Provide certificate

$$\begin{bmatrix} \dot{c_p} \\ \dot{c_1} \\ \dot{c_2} \\ \dot{c_e} \\ \dot{u} \end{bmatrix} = \begin{pmatrix} -(k_{10}+k_{12}+k_{13}) & k_{12} & k_{13} & 0 & 1/V_1 \\ k_{21} & -k_{21} & 0 & 0 & 0 \\ k_{31} & 0 & -k_{31} & 0 & 0 \\ k_d & 0 & 0 & -k_d & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} c_p \\ c_1 \\ c_2 \\ c_e \\ u \end{bmatrix}$$

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This technique assumes the model is accurate!

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Motivation – Example of Anesthesia Model

Perform analysis on this model

Safe

Unsafe

Provide certificate

This technique assumes the model is accurate! *i.e. all the values in the matrix are accurate*

$$\begin{bmatrix} \dot{c_p} \\ \dot{c_1} \\ \dot{c_2} \\ \dot{c_e} \\ \dot{u} \end{bmatrix} = \begin{pmatrix} -(k_{10}+k_{12}+k_{13}) & k_{12} & k_{13} & 0 & 1/V_1 \\ k_{21} & -k_{21} & 0 & 0 & 0 \\ k_{31} & 0 & -k_{31} & 0 & 0 \\ k_d & 0 & 0 & -k_d & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} c_p \\ c_1 \\ c_2 \\ c_e \\ u \end{bmatrix}$$

Motivation – Example of Anesthesia Model

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Analysis

Safe *(Say)*

Motivation – Example of Anesthesia Model

$$
\begin{bmatrix} \dot{c_p} \\ \dot{c_1} \\ \dot{c_2} \\ \dot{c_e} \\ \dot{u} \end{bmatrix} =
\begin{pmatrix}
-(k_{10}+k_{12}+k_{13}) & k_{12} & k_{13} & 0 & {\color{red}1/V_1} \\
k_{21} & -k_{21} & 0 & 0 & 0 \\
k_{31} & 0 & -k_{31} & 0 & 0 \\
k_d & 0 & 0 & -k_d & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{bmatrix} c_p \\ c_1 \\ c_2 \\ c_e \\ u \end{bmatrix}
$$

Modelling Error

Analysis

Safe *(Say)*

What if we discover an error in the model now?

Motivation – Example of Anesthesia Model

$$\begin{bmatrix} \dot{c_p} \\ \dot{c_1} \\ \dot{c_2} \\ \dot{c_e} \\ \dot{u} \end{bmatrix} = \begin{pmatrix} -(k_{10}+k_{12}+k_{13}) & k_{12} & k_{13} & 0 & 1/V_1 \\ k_{21} & -k_{21} & 0 & 0 & 0 \\ k_{31} & 0 & -k_{31} & 0 & 0 \\ k_d & 0 & 0 & -k_d & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} c_p \\ c_1 \\ c_2 \\ c_e \\ u \end{bmatrix}$$

Analysis

Safe (Say)

Would it still be safe?

Motivation – Example of Anesthesia Model

$$\begin{bmatrix} \dot{c_p} \\ \dot{c_1} \\ \dot{c_2} \\ \dot{c_e} \\ \dot{u} \end{bmatrix} = \begin{pmatrix} -(k_{10}+k_{12}+k_{13}) & k_{12} & k_{13} & 0 & {\color{red}1/V_1} \\ k_{21} & -k_{21} & 0 & 0 & 0 \\ k_{31} & 0 & -k_{31} & 0 & 0 \\ k_d & 0 & 0 & -k_d & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} c_p \\ c_1 \\ c_2 \\ c_e \\ u \end{bmatrix}$$

Analysis

Safe (Say)

Need to perform analysis again form scratch!

Contribution

- Class of uncertainties for which analyzing the system is efficient.
- Given a dynamical system, introduce such uncertainties and compute *Robust Reachable Set*

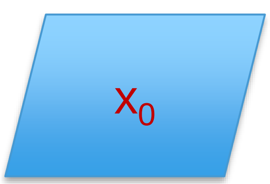
Outline

- Motivation
- Problems due to uncertainties in *verification*
- A class uncertain dynamics - with limited effect of uncertainties in the system
- Introduction of *uncertainties* in a system
- Evaluation

Background

- Trajectories: Evolution of the linear discrete system in time.

$$\xi_A(x_0, 0) = x_0$$

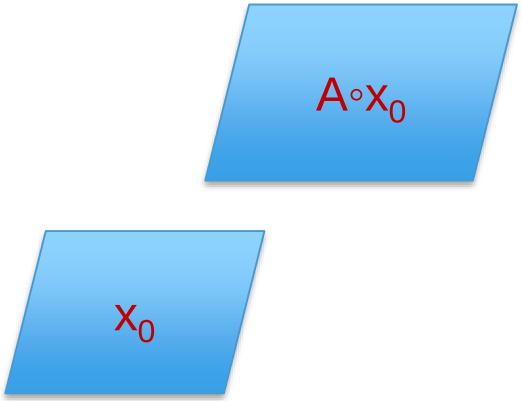


Evolution of system at time 0

Trajectories

- Trajectories: Evolution of the linear discrete system in time.

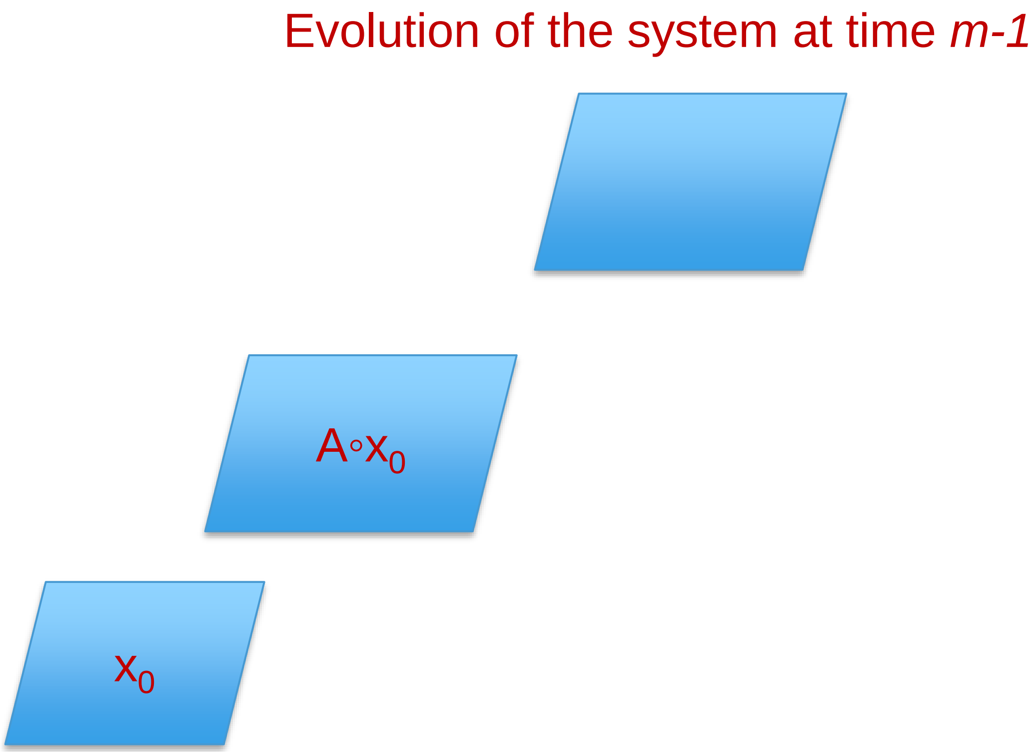
$$\xi_A(x_0, 1) = Ax_0$$



Evolution of system at time 1

Trajectories

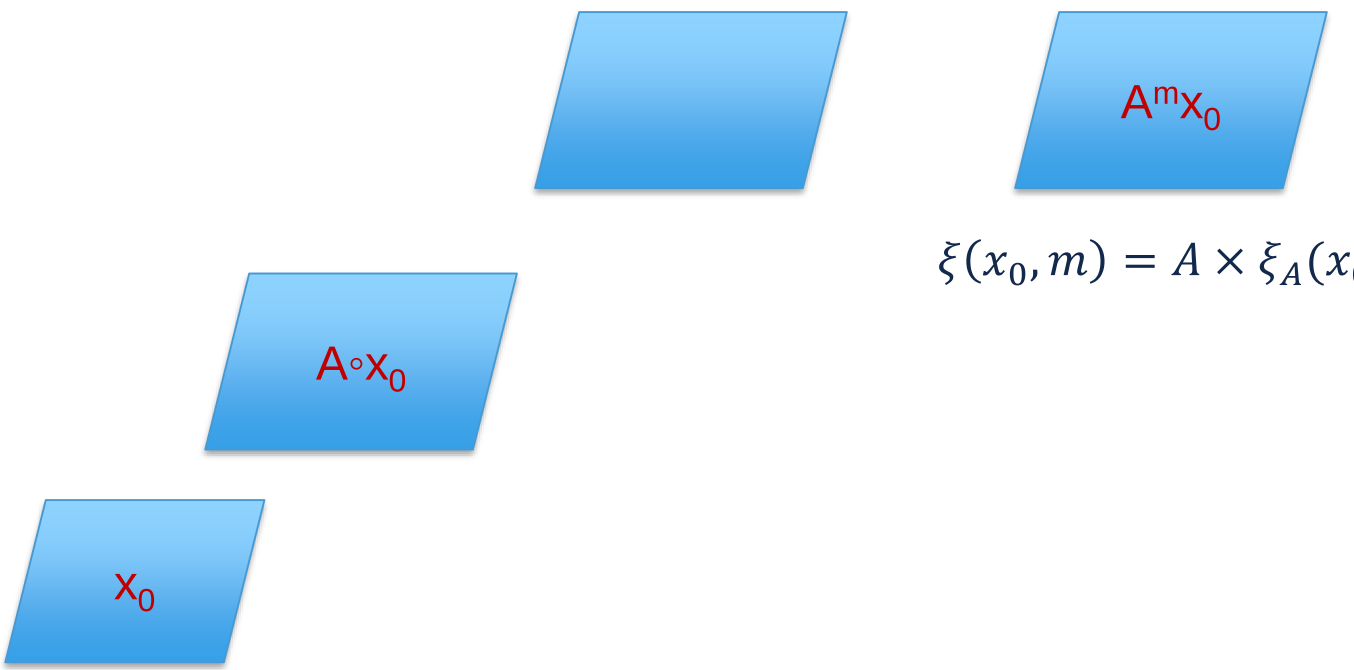
- Trajectories: Evolution of the linear discrete system in time.



Trajectories

- Trajectories: Evolution of the linear discrete system in time.

Evolution of the system at time *m*

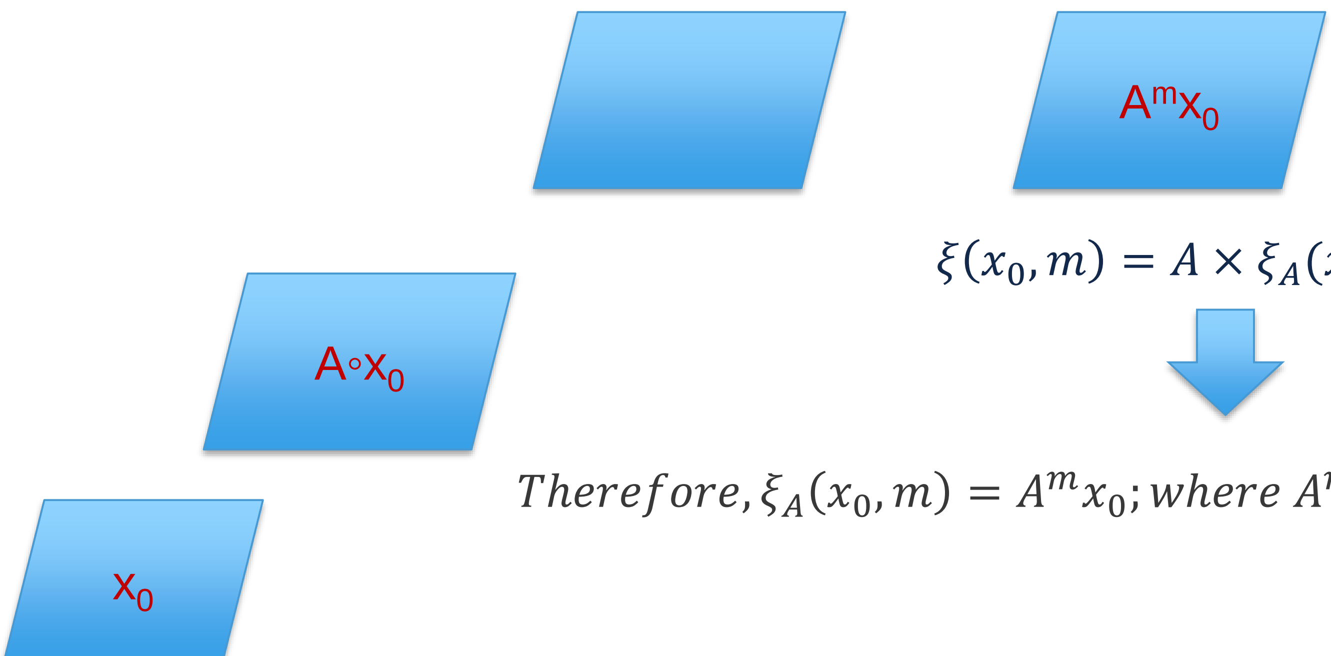


$$\xi(x_0, m) = A \times \xi_A(x_0, m-1)$$

Trajectories

- Trajectories: Evolution of the linear discrete system in time.

Evolution of the system at time *m*



$$\xi(x_0, m) = A \times \xi_A(x_0, m-1)$$

$$Therefore, \xi_A(x_0, m) = A^m x_0; where\ A^m = \underbrace{A \times A \times \cdots A}_{\text{m times}}$$

Matrix Multiplication is a crucial operation in computing trajectories

What are Linear Uncertain Systems?

Definition (Uncertain Linear Systems and Reachable Set).

$$s^+ = \Lambda s$$

$$\begin{bmatrix} s_1^+ \\ s_2^+ \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & 2 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$$

System with uncertainties

Reachable Set of Uncertain Linear Systems

$$\begin{bmatrix} s_1^+ \\ s_2^+ \end{bmatrix} = A \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$$

Where $A = \begin{bmatrix} x & y \\ 0 & 2 \end{bmatrix}$, $x \in [2,3]$ and $y \in [4,5]$

Reachable Set of Uncertain Linear Systems

$$\begin{bmatrix} s_1^+ \\ s_2^+ \end{bmatrix} = A \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$$

Where $\mathrm{A} = \begin{bmatrix} x & y \\ 0 & 2 \end{bmatrix}$, $x \in [2,3]$ and $y \in [4,5]$

At step $t = 2$, with Initial Set θ

$$\mathrm{A}^2 = \begin{bmatrix} x^2 & xy + 2y \\ 0 & 4 \end{bmatrix}$$

$$s_1^{[2]} = x^2\theta_1 + (xy + 2y)\theta_2 \,, \qquad s_2^{[2]} = 4\theta_2$$

Reachable Set of Uncertain Linear Systems

$$\begin{bmatrix} s_1^+ \\ s_2^+ \end{bmatrix} = A \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$$

Where $\mathrm{A} = \begin{bmatrix} x & y \\ 0 & 2 \end{bmatrix}$, $x \in [2,3]$ and $y \in [4,5]$

At step $t = 3$, with Initial Set θ

$$\mathrm{A}^3 = \begin{bmatrix} x^3 & x^2y + 2xy + 4y \\ 0 & 8 \end{bmatrix}$$

$$s_1^{[2]} = x^2\theta_1 + (xy + 2y)\theta_2\,, \qquad s_2^{[2]} = 4\theta_2$$

$$s_1^{[3]} = x^3\theta_1 + (x^2y + 2xy + 4y)\theta_2\,, \qquad s_2^{[3]} = 8\theta_2$$

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Uncertainty Polynomial grows with time

Uncertainties

$$A = \begin{bmatrix} \boxed{x} & \boxed{y} \\ 0 & 2 \end{bmatrix}$$

Uncertainty Polynomial grows with time

$$A = \begin{bmatrix} x & y \\ 0 & 2 \end{bmatrix}$$

Higher powers of x and *y*

$$A^2 = \begin{bmatrix} x^2 & xy + 2y \\ 0 & 2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} x^3 & x^2y + 2xy + 4y \\ 0 & 2 \end{bmatrix}$$

Problems with High Degree Polynomials

$$\begin{bmatrix} s_1^+ \\ s_2^+ \end{bmatrix} = A \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \qquad where\ A = \begin{bmatrix} x & y \\ 0 & 2 \end{bmatrix}, x \in [2,3], y \in [4,5]$$

Safety Condition

$The\ system\ is\ considered\ to\ be\ safe\ if\ at\ every\ step\ the\ value\ of\ s_1 < 100$

Problems with High Degree Polynomials

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Safety check at step 5

$$A^5 = \begin{bmatrix} f_1(x,y) & f_2(x,y) \\ 0 & 32 \end{bmatrix}$$

Where, $f_1(x,y) = x^5$

$$f_2(x,y) = x^4 + 2x^2y + 4xy + 8y$$

Check: $f_1(x,y)\theta_1 + f_2(x,y)\theta_2 \geq 100$

θ_i are the constraints on s_i in the initial set

Problems with High Degree Polynomials

$$\begin{bmatrix} s_1^+ \\ s_2^+ \end{bmatrix} = A \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$$

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Check involving very high degree polynomials

Problems with High Degree Polynomials

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As step size increases, checking becomes infeasible!

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- **A class uncertain dynamics - with limited effect of uncertainties in the system (*linearity of uncertain variables*)**
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A class of uncertainties for which the polynomial does not have higher order terms

$$A = \begin{bmatrix} 1 & \alpha \\ 0 & 2 \end{bmatrix}$$

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$$A = \begin{bmatrix} 1 & \alpha \\ 0 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 3\alpha \\ 0 & 4 \end{bmatrix}$$

A class of uncertainties for which the polynomial does not have higher order terms

$$A = \begin{bmatrix} 1 & \alpha \\ 0 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 3\alpha \\ 0 & 4 \end{bmatrix}$$

A^m does not contain any higher order term of α, for all m

A class of uncertainties for which the polynomial does not have higher order terms

$$\begin{bmatrix} s_1^+ \\ s_2^+ \end{bmatrix} = A \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \qquad where\ A = \begin{bmatrix} 1 & \alpha \\ 0 & 2 \end{bmatrix}, \alpha \in [2,3]$$

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$$A^5 = \begin{bmatrix} 1 & 31\alpha \\ 0 & 32 \end{bmatrix}$$

Check: $\theta_1 + 31\alpha\theta_2 \geq 100$

θ_i are the constraints on s_i in the initial set

A class of uncertainties for which the polynomial does not have higher order terms

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Safety check at step 5

$$A^5 = \begin{bmatrix} 1 & 31\alpha \\ 0 & 32 \end{bmatrix}$$

Check: $\theta_1 + 31\alpha\theta_2 \geq 100$

Check involving **bi-linear** constraints

A class of uncertainties for which the polynomial does not have higher order terms

$$\begin{bmatrix} s_1^+ \\ s_2^+ \end{bmatrix} = A \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \qquad where\ A = \begin{bmatrix} 1 & \alpha \\ 0 & 2 \end{bmatrix}, \alpha \in [2,3]$$

Safety Condition

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Safety check at step 5

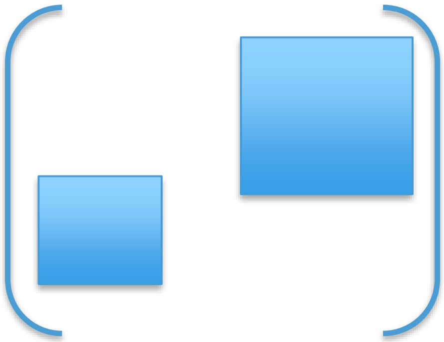
$$A^5 = \begin{bmatrix} 1 & 31\alpha \\ 0 & 32 \end{bmatrix}$$

Check: $\theta_1 + 31\alpha\theta_2 \geq 100$

This is observed at all steps!

A class of uncertainties for which the polynomial does not have higher order terms

- How to characterize such uncertainties?



Linear Dynamics

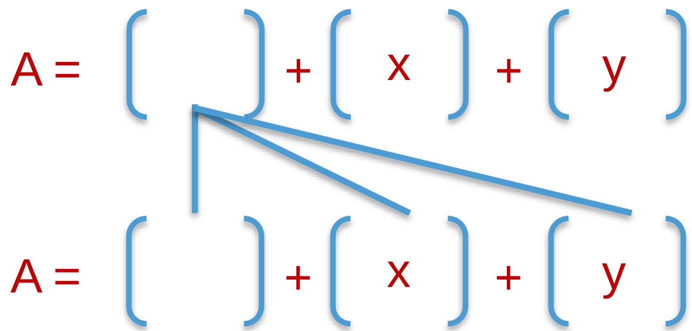
Sufficient conditions based on the structure of the matrix, ensuring no higher order terms

Uncertain Systems Using Linear Matrix Expression (LME)

$$\begin{pmatrix} 1+x & y & 4 \\ 2x+y & 8 & 3 \\ 1 & y & x \end{pmatrix} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 8 & 3 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \mathrm{y}$$

Represented using coefficient matrices

Matrix Exponents



Matrix Exponents

$$A = [\ \ ] + [\,x\,] + [\,y\,]$$

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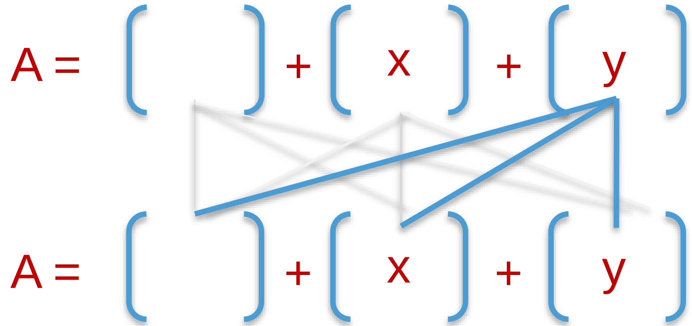
Matrix Exponents



Matrix Exponents

$$A = [\ \ ] + [x] + [y]$$

$$A = [\ \ ] + [x] + [y]$$

$$A \times A = [\ \ ] + [x] + [y] + [z] + [x^2] + \cdots + [xy]$$

Matrix Exponents

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$$A \times A^2 = [\ \ ] + [x] + [y] + [z] + [x^3] + \cdots + [xy]$$

Matrix Exponents

$$A = \left[\ \ \right] + \left[\ x\ \right] + \left[\ y\ \right]$$

$$A = \left[\ \ \right] + \left[\ x\ \right] + \left[\ y\ \right]$$

$$A \times A = \left[\ \ \right] + \left[\ x\ \right] + \left[\ y\ \right] + \left[\ z\ \right] + \left[\ x^2\ \right] + \cdots + \left[\ xy\ \right]$$

$$A \times A^2 = \left[\ \ \right] + \left[\ x\ \right] + \left[\ y\ \right] + \left[\ z\ \right] + \left[\ x^3\ \right] + \cdots + \left[\ xy\ \right]$$

$$\vdots$$

$$A \times A^{m-1} = \left[\ \ \right] + \left[\ x\ \right] + \left[\ y\ \right] + \left[\ z\ \right] + \left[\ x^m\ \right] + \cdots + \left[\ xy\ \right]$$

Matrix Exponents

$$A = [\;\;] + [\,x\,] + [\,y\,]$$

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$$A \times A = [\;\;] + [\,x\,] + [\,y\,] + [\,z\,] + [\,x^2\,] + \cdots + [\,xy\,]$$

A^i will have i-th order terms of the uncertain variables

$$A \times A^{m-1} = [\;\;] + [\,x\,] + [\,y\,] + [\,z\,] + [\,x^m\,] + \cdots + [\,xy\,]$$

Matrix Exponents

$$A = \begin{bmatrix} \; \end{bmatrix} + \begin{bmatrix} x \end{bmatrix} + \begin{bmatrix} y \end{bmatrix}$$

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How do we ensure Linear Matrix Expression (LME) at all steps?

$$A \times A^{m-1} = \begin{bmatrix} \; \end{bmatrix} + \begin{bmatrix} x \end{bmatrix} + \begin{bmatrix} y \end{bmatrix} + \begin{bmatrix} z \end{bmatrix} + \begin{bmatrix} x^m \end{bmatrix} + \cdots + \begin{bmatrix} xy \end{bmatrix}$$

When are Linear Matrix Expressions (LME) closed under multiplication?

$$A = \begin{bmatrix} \; \end{bmatrix} + \begin{bmatrix} x \end{bmatrix} + \begin{bmatrix} y \end{bmatrix}$$

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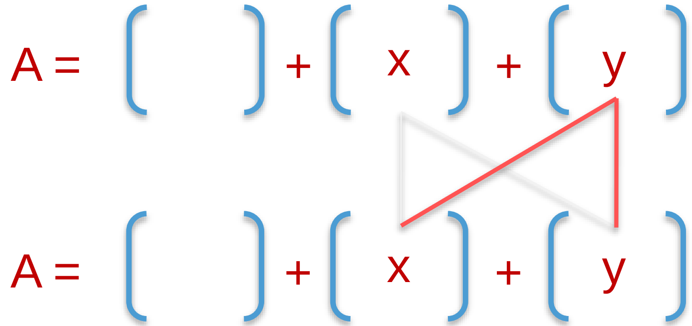
Interaction of *x* with itself and others

Produces higher order terms

When are Linear Matrix Expressions (LME) closed under multiplication?



Interaction of *y* with itself and others

Produces higher order terms

When are Linear Matrix Expressions (LME) closed under multiplication?

$$A = [\quad] + [\; x \;] + [\; y \;]$$

$$A = [\quad] + [\; x \;] + [\; y \;]$$

$$A^2 = [\quad] + [\; x \;] + [\; y \;]$$

If these interactions are 0 then the product is closed

Linear Matrix Expression (LME) Exponents

$$A = \left[\quad\right] + \left[\ x\ \right] + \left[\ y\ \right]$$

Assume: A × A is closed

Linear Matrix Expression (LME) Exponents

$$A = \begin{bmatrix} \quad \end{bmatrix} + \begin{bmatrix} x \end{bmatrix} + \begin{bmatrix} y \end{bmatrix}$$

$$A^2 = \begin{bmatrix} \quad \end{bmatrix} + \begin{bmatrix} x \end{bmatrix} + \begin{bmatrix} y \end{bmatrix}$$

Linear Matrix Expression (LME) Exponents

$$A = \left(\quad \right) + \left(x \right) + \left(y \right)$$

$$A^2 = \left(\quad \right) + \left(x \right) + \left(y \right)$$

What about A^3

Linear Matrix Expression (LME) Exponents

$$A = [\;\;] + [\;x\;] + [\;y\;]$$

$$A^2 = [\;\;] + [\;x\;] + [\;y\;]$$

Assume: A × A is closed

This assumption is not enough

Not guaranteed to be an LME

$$A^3 = [\;\;] + [\;x\;] + [\;y\;] + [\;x^2\;] + \cdots + [\;xy\;]$$

Linear Matrix Expression (LME) Exponents

$$A = [\quad] + [x] + [y]$$

$$A^2 = [\quad] + [x] + [y]$$

Not guaranteed to be an LME

$$A^3 = [\quad] + [x] + [y] + [x^2] + \cdots + [xy]$$

$$A^4 = [\quad] + [x] + [y] + [x^3] + \cdots + [xy]$$

Linear Matrix Expression (LME) Exponents

$$A = \left[\quad\right] + \left[\; x \;\right] + \left[\; y \;\right]$$

$$A^2 = \left[\quad\right] + \left[\; x \;\right] + \left[\; y \;\right]$$

Need to ensure that the product is an LME at every step!!

$$A^3 = \left[\quad\right] + \left[\; x \;\right] + \left[\; y \;\right] + \left[\; x^2 \;\right] + \cdots + \left[\; xy \;\right]$$

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Outline

- Motivation
- Problems due to uncertainties in *verification*
- **A class uncertain dynamics - with limited effect of uncertainties in the system (*linearity of uncertain variables*)**
- Introduction of *uncertainties* in a system
- Evaluation

Matrix Support

To represent the structure of a matrix

↓

Matrix Support

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↓

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Boolean Abstraction of matrix, that distinguishes between zero and non-zero elements

Matrix Support

To represent the structure of a matrix

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Matrix Support

Boolean Abstraction of matrix, that distinguishes between zero and non-zero elements

We perform the operations on Matrix Supports instead of performing them on actual matrices

The conditions will be imposed on Matrix Supports and not actual matrices

Matrix Support

Matrix Support

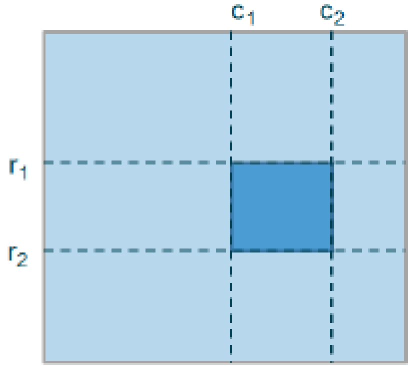
$$supp\left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathbf{2} & \mathbf{-2} & 0 \\ 0 & \mathbf{-3} & \mathbf{4} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix Support

Matrix Support

$$supp\left(\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & \mathbf{2} & \mathbf{-2} & 0\\ 0 & \mathbf{-3} & \mathbf{4} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\right)=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & \mathbf{1} & \mathbf{1} & 0\\ 0 & \mathbf{1} & \mathbf{1} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

If the support have a nice form as above:



Pictorially:
Light blue: 0
Dark Blue: 1

Matrix Support

Matrix Support

$$supp\left(\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & \mathbf{2} & \mathbf{-2} & 0\\ 0 & \mathbf{-3} & \mathbf{4} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\right) = \begin{bmatrix}0 & 0 & 0 & 0\\ 0 & \mathbf{1} & \mathbf{1} & 0\\ 0 & \mathbf{1} & \mathbf{1} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

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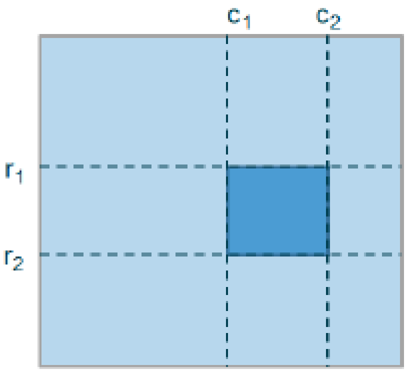

Pictorially:
Light blue: 0
Dark Blue: 1

$\rightarrow$ $block((r_1, c_1), (r_2, c_2))$

Matrix Support

Sub Support and Super Support

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{0} & 0 \\ 0 & \mathbf{0} & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \leq \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$B_1 \leq B_2$

$$if\ B_2[i,j] = 0\ then\ B_1[i,j] = 0$$

Matrix Support

Addition of Supports

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Same as matrix addition
- Logical OR instead of +

$$B_3[i,j] = B_1[i,j] \vee B_2[i,j]$$

Matrix Support

Addition of Supports

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \oplus \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Same as matrix addition
- Logical OR instead of +

$$B_3[i,j] = B_1[i,j] \vee B_2[i,j]$$

Multiplication of Supports

Same as matrix multiplication
- Logical OR instead of +
- Logical AND instead of ×

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \otimes \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}(1 \wedge 0) \vee (0 \wedge 0) & (1 \wedge 0) \vee (0 \wedge 1)\\(0 \wedge 0) \vee (1 \wedge 0) & (0 \wedge 0) \vee (1 \wedge 1)\end{bmatrix}$$

$$B_3[i,j] = \bigvee_{l=1}^{k} B_1[i,l] \wedge B_2[l,j]$$

Properties of Support

- $\sup\left(\begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\right) \leq \sup\left(\begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}\right) \oplus \sup\left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\right)$

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- $\sup\left(\begin{bmatrix}2 & 0\\3 & -1\end{bmatrix}\times\begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}\right) \leq \sup\left(\begin{bmatrix}2 & 0\\3 & -1\end{bmatrix}\right)\otimes\sup\left(\begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}\right)$

Given matrices M_1 *and* M_2, *if* $\mathrm{supp}(M_1)\otimes\mathrm{supp}(M_2)=\mathbf{0}$, *then* $M_1\times M_2=\mathbf{0}$

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- $\sup\left(\begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\right) \leq \sup\left(\begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}\right) \otimes \sup\left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\right)$

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- $\begin{bmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{bmatrix} \leq \begin{bmatrix} \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{1} \end{bmatrix}$

Properties of Support

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Sufficient Conditions for Representing Reach Set of Uncertain Systems

If $\forall i, j, 1 \leq i \leq k$ *and* $1 \leq j \leq k$, $\operatorname{supp}(N_i) \otimes \operatorname{supp}(M_j) = \boldsymbol{0}$, *then* $A \times B$ *results in an LME.*

$$\begin{bmatrix} \; N_0 \; \end{bmatrix} + \begin{bmatrix} \; N_1 \; \end{bmatrix} x + \begin{bmatrix} \; N_2 \; \end{bmatrix} y$$

$$\begin{bmatrix} \; M_0 \; \end{bmatrix} + \begin{bmatrix} \; M_1 \; \end{bmatrix} x + \begin{bmatrix} \; M_2 \; \end{bmatrix} y$$

Sufficient Conditions for Representing Reach Set of Uncertain Systems

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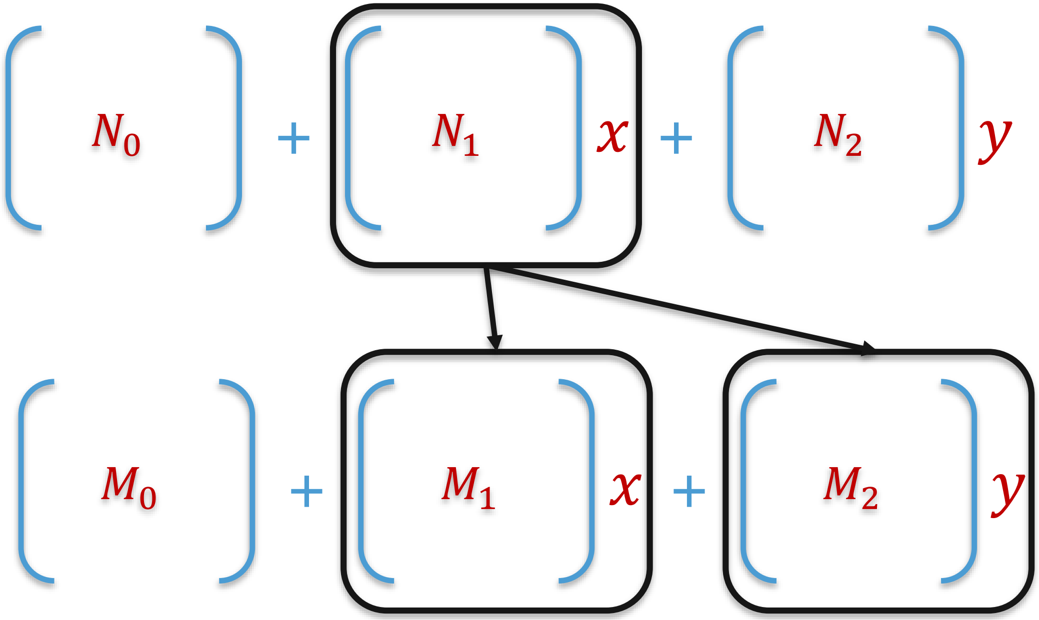

IF product of their supports are 0

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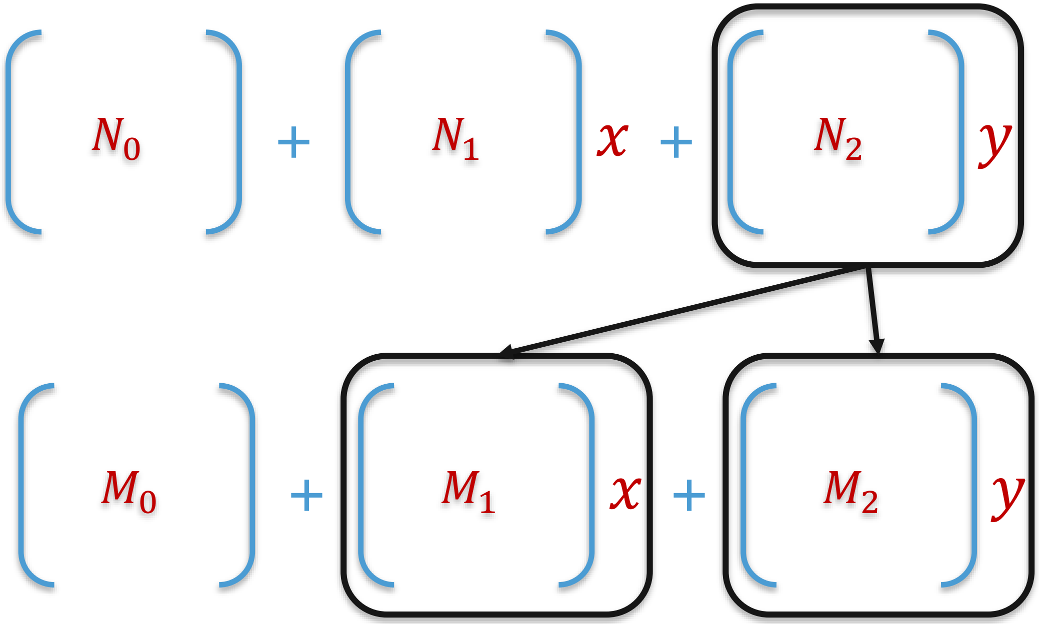

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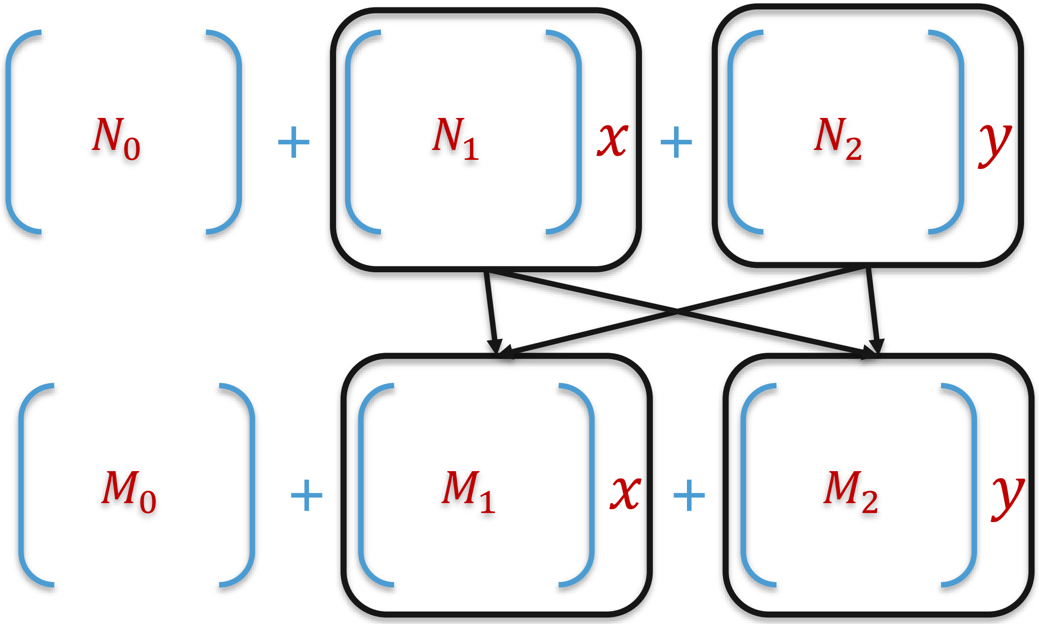

Then from the properties of support, the product will be an LME

Recap - LME Exponents

$A = [\ \ ] + [\ x\ ] + [\ y\ ]$

$A^2 = [\ \ ] + [\ x\ ] + [\ y\ ]$

Assume: A × A is closed

This assumption is not enough

Not guaranteed to be an LME

$A^3 = [\ \ ] + [\ x\ ] + [\ y\ ] + [\ x^2\ ] + \cdots + [\ xy\ ]$

Recap - LME Exponents

$$A = \left[\quad \right] + \left[x \right] + \left[y \right]$$

$$A^2 = \left[\quad \right] + \left[x \right] + \left[y \right]$$

Need to ensure that the product is an LME at every step!!

$$A^3 = \left[\quad \right] + \left[x \right] + \left[y \right] + \left[x^2 \right] + \cdots + \left[xy \right]$$

$$A^4 = \left[\quad \right] + \left[x \right] + \left[y \right] + \left[x^3 \right] + \cdots + \left[xy \right]$$

Sufficient Conditions for Representing Reach Set of Uncertain Systems

if

$$\forall i, j, 1 \leq i, j \leq k, \text{supp}(N_i) \otimes \text{supp}(N_j) = \boldsymbol{0}$$ Cond. 1

$$\forall i, 0 \leq i \leq k, \text{supp}(N_0) \otimes \text{supp}(N_i) \leq \text{supp}(N_i), \quad \textit{and}\ \text{supp}(N_i) \otimes \text{supp}(N_0) \leq \text{supp}(N_i).$$ Cond. 2

then for all $m \geq 2$, A^m *is a Linear Matrix Expression (LME)*

$$\left(N_0 \right) + \left(N_1 \right) x + \left(N_2 \right) y$$

Sufficient Conditions for Representing Reach Set of Uncertain Systems

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$$\forall i, j, 1 \le i, j \le k, \mathrm{supp}(N_i) \otimes \mathrm{supp}(N_j) = \boldsymbol{0}$$

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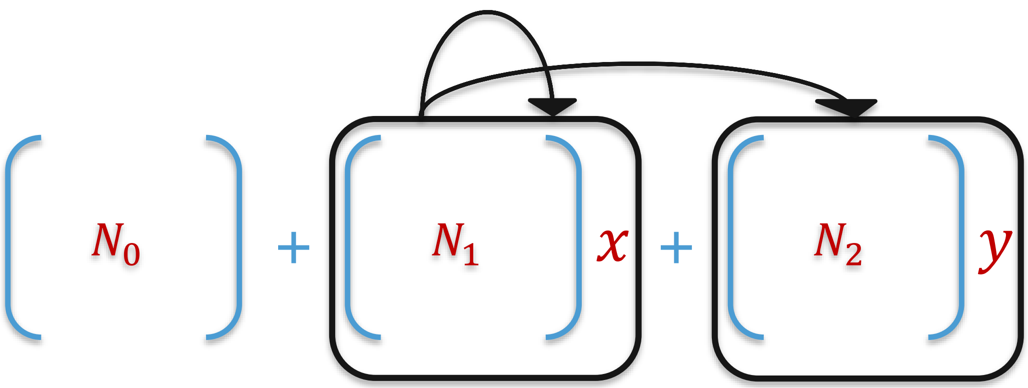
Product of their Supports be 0



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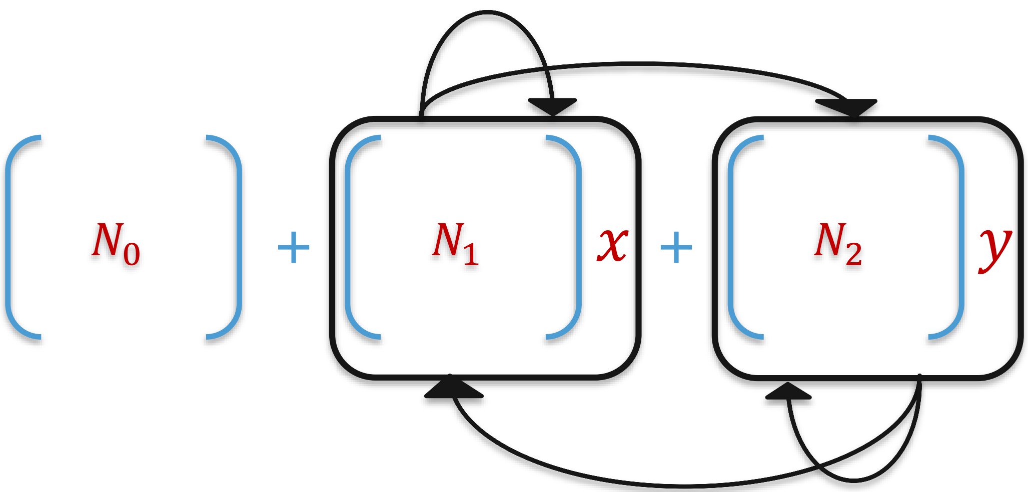

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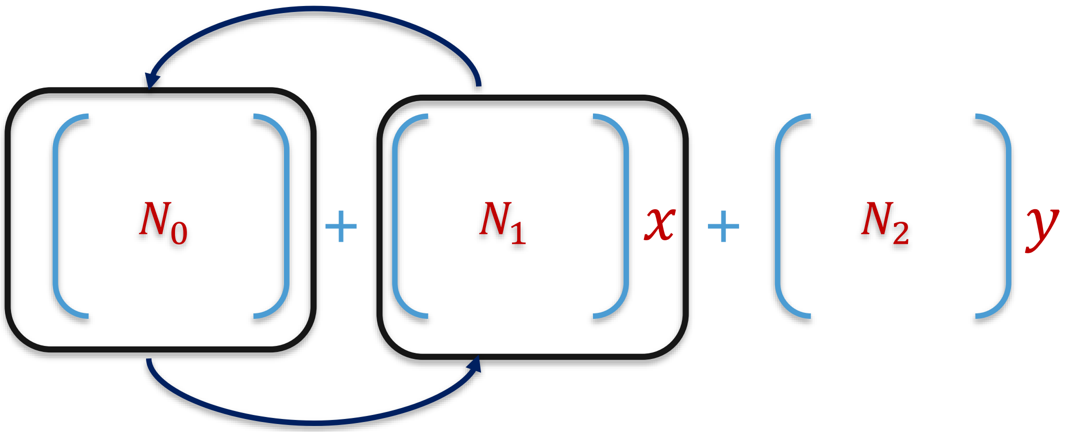

Product of their supports should be a sub-support of supp(N_1)

Sufficient Conditions for Representing Reach Set of Uncertain Systems

if

$$\forall i, j, 1 \leq i, j \leq k, \text{supp}(N_i) \otimes \text{supp}(N_j) = \boldsymbol{0}$$

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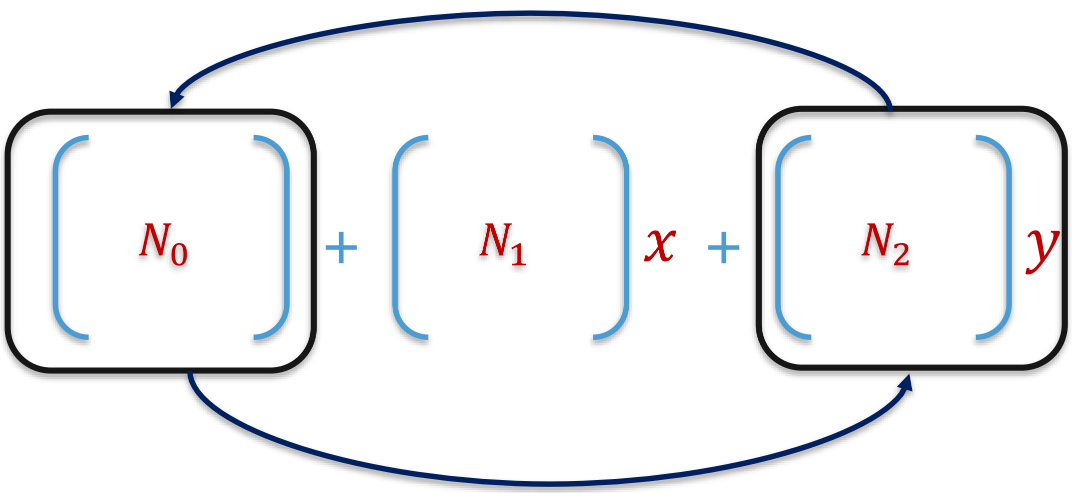
$then\ for\ all\ m \geq 2, A^m is\ a\ Linear\ Matrix\ Expression\ (LME)$



Product of their supports should be a sub-support of supp(N_2)

Sufficient Conditions for Representing Reach Set of Uncertain Systems

if

$$\forall i, j, 1 \leq i, j \leq k, \text{supp}(N_i) \otimes \text{supp}(N_j) = \mathbf{0}$$

$$\forall i, 0 \leq i \leq k, \text{supp}(N_0) \otimes \text{supp}(N_i) \leq \text{supp}(N_i),$$
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$then\ for\ all\ m \geq 2, A^m is\ a\ Linear\ Matrix\ Expression\ (LME)$

$A = \quad N_0 \quad N_1 \quad N_2 \quad \cdots \quad N_k$

Sufficient Conditions for Representing Reach Set of Uncertain Systems

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Ensures A^2 will be an LME

then for all $m \geq 2$, A^m *is a Linear Matrix Expression (LME)*

$A =$ N_0 N_1 N_2 $\cdots$ N_k

$A^2 =$ $N_0^{[2]}$ $N_1^{[2]}$ $N_2^{[2]}$ $\cdots$ $N_k^{[2]}$

Sufficient Conditions for Representing Reach Set of Uncertain Systems

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$$\forall i, j, 1 \le i, j \le k, \mathrm{supp}(N_i) \otimes \mathrm{supp}(N_j) = \boldsymbol{0}$$

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$$A = supp(N_0) \quad supp(N_1) \quad supp(N_2) \quad \cdots \quad supp(N_k)$$

$$A^2 = supp(N_0^{[2]}) \quad supp(N_1^{[2]}) \quad supp(N_2^{[2]}) \quad \cdots \quad supp(N_k^{[2]})$$

Sufficient Conditions for Representing Reach Set of Uncertain Systems

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$\forall i, 0 \le i \le k, \text{supp}(N_0) \otimes \text{supp}(N_i) \le \text{supp}(N_i),$
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Ensures sub-support behavior

$then\ for\ all\ m \ge 2, A^m is\ a\ Linear\ Matrix\ Expression\ (LME)$

$A = \quad supp(N_0) \qquad supp(N_1) \qquad supp(N_2) \qquad \cdots \qquad supp(N_k)$

Sub-support

$A^2 = \quad supp(N_0^{[2]}) \qquad supp(N_1^{[2]}) \qquad supp(N_2^{[2]}) \qquad \cdots \qquad supp(N_k^{[2]})$

Sufficient Conditions for Representing Reach Set of Uncertain Systems

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By Induction

$then\ for\ all\ m \ge 2, A^m is\ a\ Linear\ Matrix\ Expression\ (LME)$

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$$\vdots$$

$$A^m = \quad supp(N_0^{[m]}) \quad supp(N_1^{[m]}) \quad supp(N_2^{[m]}) \quad \cdots \quad supp(N_k^{[m]})$$

Sufficient Conditions for Representing Reach Set of Uncertain Systems

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$then\ for\ all\ m \geq 2, A^m is\ an\ LME$

Using this, we can sufficiently conclude that there will be no higher order terms of uncertain variables!

Sufficient Conditions for Representing Reach Set of Uncertain Systems

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$$\forall i, j, 1 \leq i, j \leq k, \text{supp}(N_i) \otimes \text{supp}(N_j) = \mathbf{0}$$

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$then \ for \ all \ m \geq 2, A^m is \ an \ LME$

We just need to check these conditions once.

And for all m, A^m will be an LME

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$then\ for\ all\ m \geq 2, A^m is\ an\ LME$

We just need to check these conditions once.

And for all m, A^m will be an LME

If A satisfies these conditions

A = [ ] + [x] + [y] + [z]

Just a one time static check!

Sufficient Conditions for Representing Reach Set of Uncertain Systems

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$$\forall i, 0 \le i \le k, \mathrm{supp}(N_0) \otimes \mathrm{supp}(N_i) \le \mathrm{supp}(N_i),$$
$$\textit{and}\ \mathrm{supp}(N_i) \otimes \mathrm{supp}(N_0) \le \mathrm{supp}(N_i).$$

then $\underline{\textit{for all}\ m \ge} 2, A^m$ *is an LME*

We just need to check these conditions once.

And for all m, A^m will be an LME

If A satisfies these conditions

$$A = (\quad) + (x) + (y) + (z)$$

$$A^2 = (\quad) + (x) + (y) + (z)$$

$$\vdots$$

$$A^m = (\quad) + (x) + (y) + (z)$$

Then A^m, will be an LME for all m

Safety Verification Algorithm of Uncertain Linear Systems

Input:

Uncertain Linear System $A = \langle N_0, N_1, \cdots, N_k \rangle$

Initial Set: Θ

Unsafe Set: U

Output:

Safe, Un-safe, Invalid

```
if Cond. 1 and Cond. 2 are satisfied:
    Compute A^i = ⟨L_0, L_1, ⋯, L_k⟩
    Compute Reachable Set RS
    if RS ∩ U = ∅
        return Safe
    else
        return Unsafe
else
    return Invalid
```

Safety Verification Algorithm of Uncertain Linear Systems

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if Cond. 1 and Cond. 2 are satisfied:
    Compute A^i = ⟨L_0, L_1, ⋯, L_k⟩
    Compute Reachable Set RS
    if RS ∩ U = ∅
        return Safe
    else
        return Unsafe
else
    return Invalid
```

Ensures A^i will be an LME for all i

Safety Verification Algorithm of Uncertain Linear Systems

Input:

Uncertain Linear System $A = \langle N_0, N_1, \cdots, N_k \rangle$
Initial Set: Θ
Unsafe Set: U

Output:

Safe, Un-safe, Invalid

```
if Cond. 1 and Cond. 2 are satisfied:
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Compute A^i

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```

Compute Reachable Set based on A^i, step, and Θ

Safety Verification Algorithm of Uncertain Linear Systems

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Initial Set: Θ

Unsafe Set: U

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    else
        return Unsafe
else
    return Invalid
```

Intersection checking is formulated as **bi-linear program** and solved using **Gurobi**

Safety Verification Algorithm of Uncertain Linear Systems

Input:

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Initial Set: Θ

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Output:

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```
if Cond. 1 and Cond. 2 are satisfied:
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    Compute Reachable Set RS
    if RS ∩ U = ∅
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    else
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else
    return Invalid
```

No intersection with unsafe set

Safety Verification Algorithm of Uncertain Linear Systems

Input:

Uncertain Linear System $A = \langle N_0, N_1, \cdots, N_k \rangle$
Initial Set: Θ
Unsafe Set: U

Output:

Safe, Un-safe, Invalid

```
if Cond. 1 and Cond. 2 are satisfied:
    Compute A^i = ⟨L_0, L_1, ⋯, L_k⟩
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    if RS ∩ U = ∅
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else
    return Invalid
```

Intersection with unsafe set

Safety Verification Algorithm of Uncertain Linear Systems

Input:

Uncertain Linear System $A = \langle N_0, N_1, \cdots, N_k \rangle$
Initial Set: Θ
Unsafe Set: U

Output:

Safe, Un-safe, Invalid

```
if Cond. 1 and Cond. 2 are satisfied:
    Compute A^i = ⟨L_0, L_1, ⋯, L_k⟩
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    if RS ∩ U = ∅
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    else
        return Unsafe
else
    return Invalid
```

Outline

- Motivation
- Problems due to uncertainties in *verification*
- A class uncertain dynamics - with limited effect of uncertainties in the system
- **Introduction of *uncertainties* in a system**
- Evaluation

Introducing Perturbations in the Linear Dynamics

Cond. 1 and *Cond. 2* are fairly restrictive!

Introducing Perturbations in the Linear Dynamics

Cond. 1 and *Cond. 2* are fairly restrictive!

Not all positions of uncertainties can satisfy these condition

$$\begin{bmatrix} \boxed{3} & 2.9 & 3.9 & \boxed{2} \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Does **NOT** Satisfy *Cond* (1) and (2)

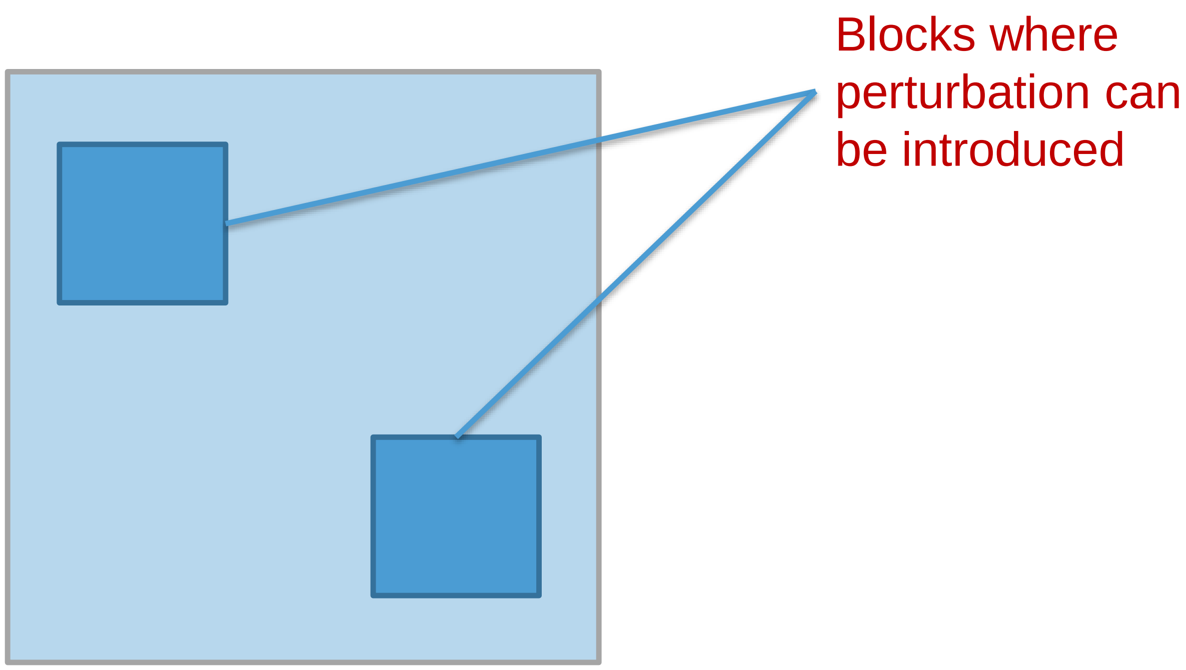
$$\begin{bmatrix} 3 & \boxed{2.9} & \boxed{3.9} & 2 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Satisfies *Cond* (1) and (2)

Introducing Perturbations in the Linear Dynamics

Cond. 1 and *Cond. 2* are fairly restrictive!

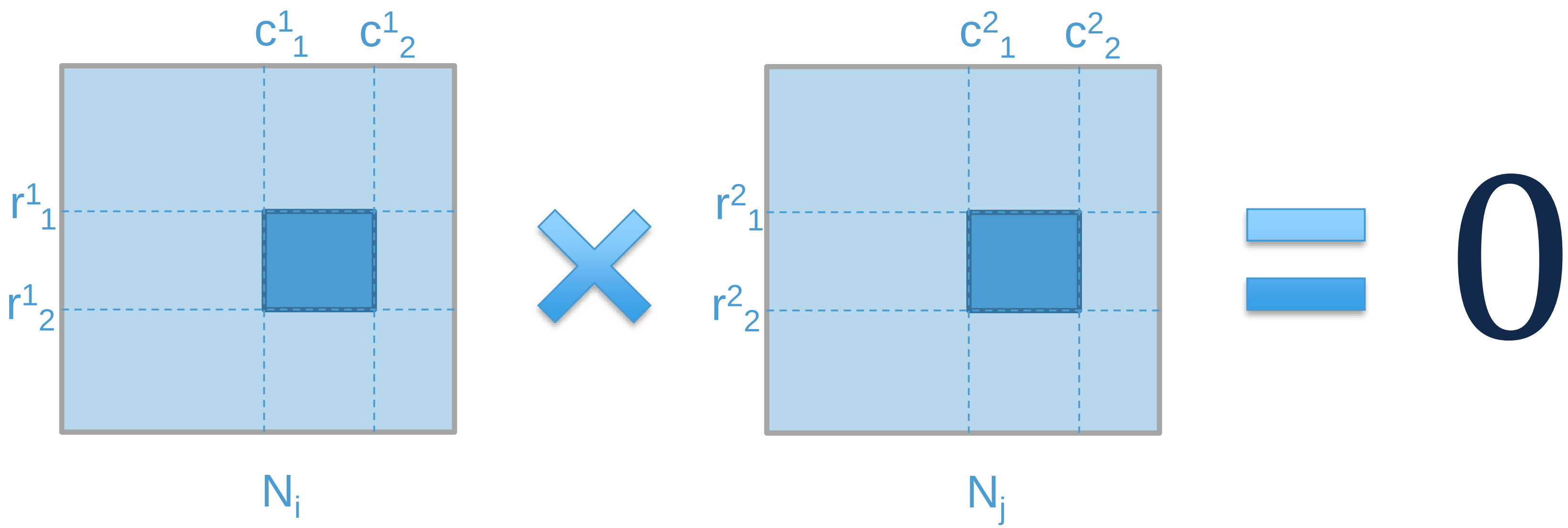
Given a linear dynamics, we introduce uncertainties such that it satisfies *conditions 1 and 2*.



Introducing Perturbations in the Linear Dynamics

- Look for all Block matrices that satisfy *Cond.* 1

$$\forall i, j, 1 \leq i, j \leq k, \text{supp}(N_i) \otimes \text{supp}(N_j) = \boldsymbol{0}$$



Introducing Perturbations in the Linear Dynamics

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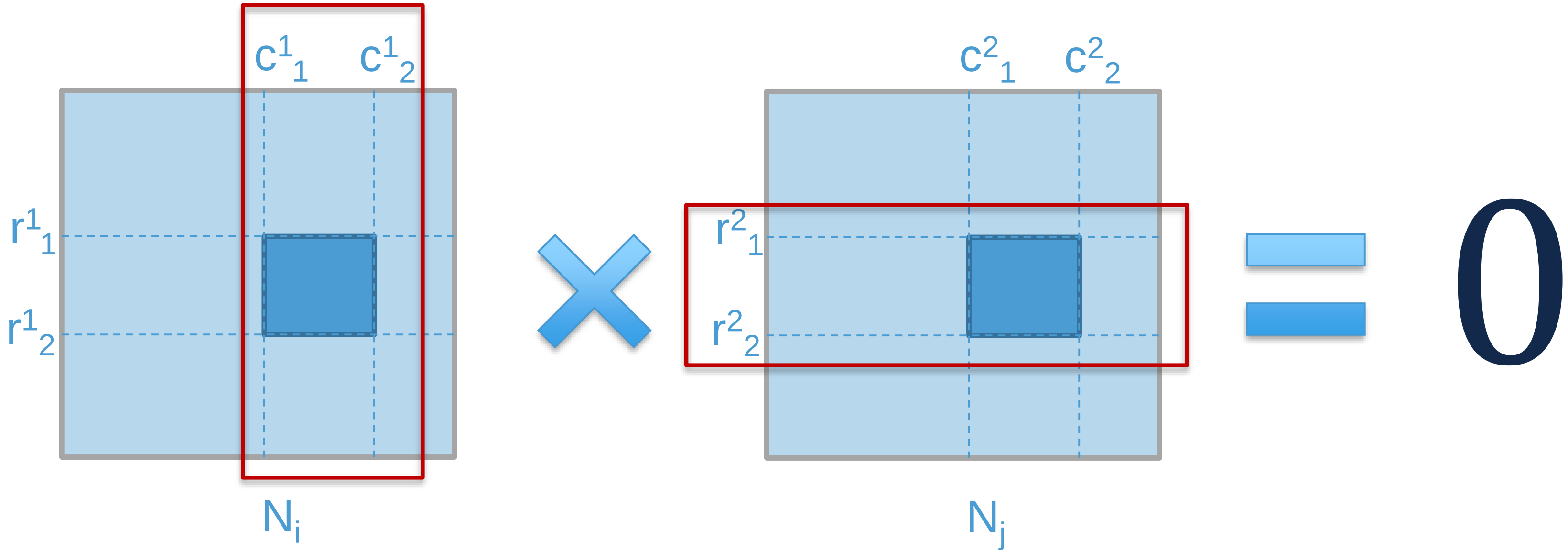


$$\{c_1^1, c_1^1 + 1, \cdots, c_2^1\} \cap \{r_1^2, r_1^2 + 1, \cdots, r_2^2\} = \emptyset$$

Introducing Perturbations in the Linear Dynamics

- Look for all Block matrices that satisfy *Cond.* 1

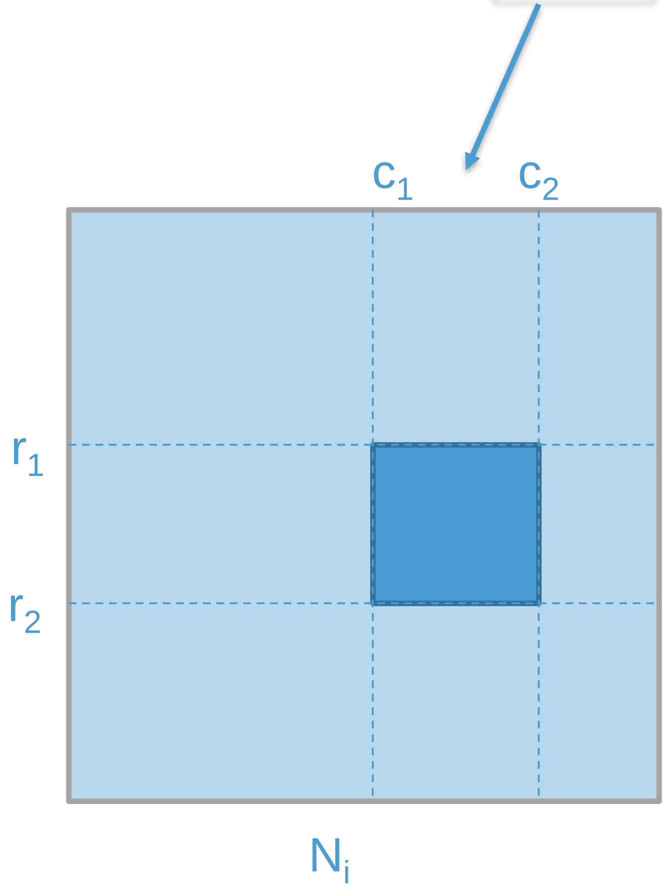
$$\forall i, j, 1 \leq i, j \leq k, \operatorname{supp}(N_i) \otimes \operatorname{supp}(N_j) = \boldsymbol{0}$$

- How to ensure the *Cond.* 2?

$$\forall i, 0 \leq i \leq k, \operatorname{supp}(N_0) \otimes \operatorname{supp}(N_i) \leq \operatorname{supp}(N_i), \\ \textit{and} \operatorname{supp}(N_i) \otimes \operatorname{supp}(N_0) \leq \operatorname{supp}(N_i).$$

Ensuring The Second Condition

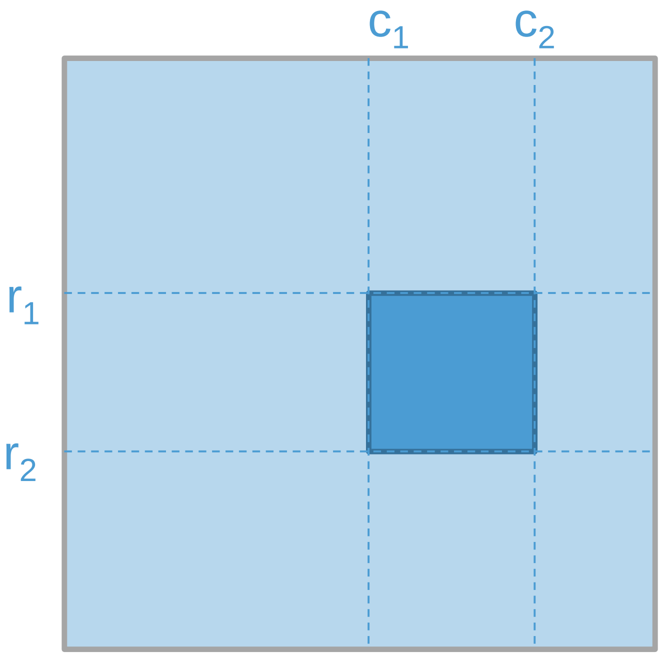
$$\forall i, 0 \le i \le k, \;\; \textit{if}\; \mathsf{supp}(N_0) \times \mathsf{supp}(N_i) \le \mathsf{supp}(N_i),$$
$$\textit{and}\; \boxed{\mathsf{supp}(N_i)} \times \mathsf{supp}(N_0) \le \mathsf{supp}(N_i).$$



Ensuring The Second Condition

$$\forall i, 0 \leq i \leq k, \ \ \textit{if}\ \mathsf{supp}(N_0) \times \mathsf{supp}(N_i) \leq \mathsf{supp}(N_i),$$
$$\textit{and}\ \mathsf{supp}(N_i) \times \mathsf{supp}(N_0) \leq \mathsf{supp}(N_i).$$

Initialize with 1



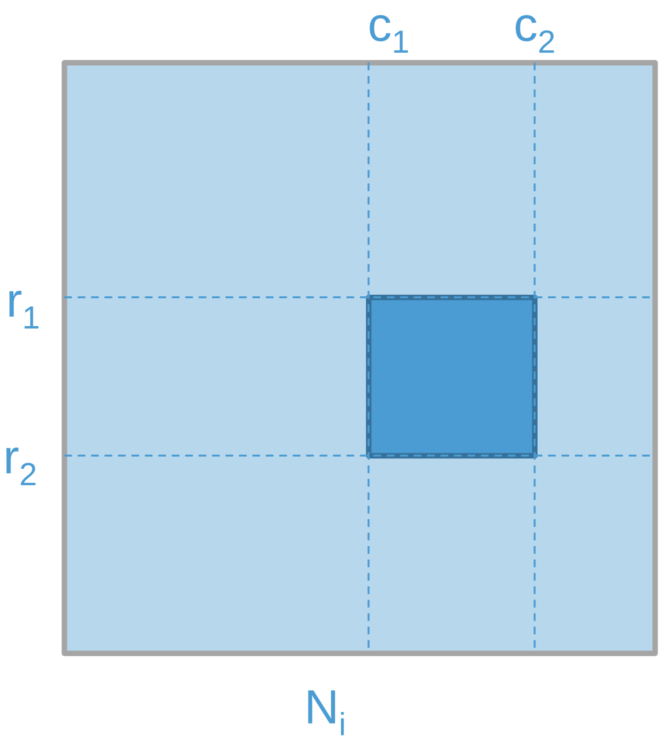
N_i

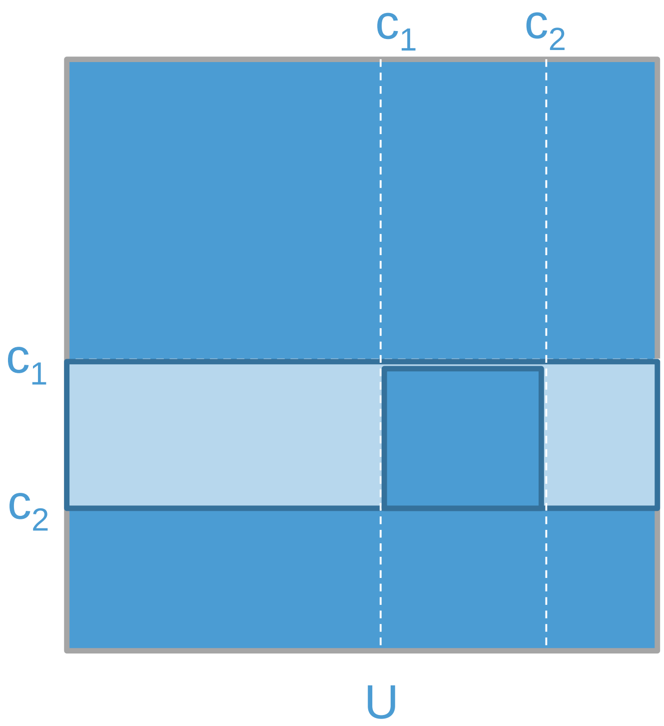
U

Ensuring The Second Condition

$$\forall i, 0 \leq i \leq k,\ \ \textit{if}\ \mathsf{supp}(N_0) \times \mathsf{supp}(N_i) \leq \mathsf{supp}(N_i),$$
$$\textit{and}\ \mathsf{supp}(N_i) \times \mathsf{supp}(N_0) \leq \mathsf{supp}(N_i).$$

Put 0s in rows c_1 to c_2 except cols c_1 and c_2

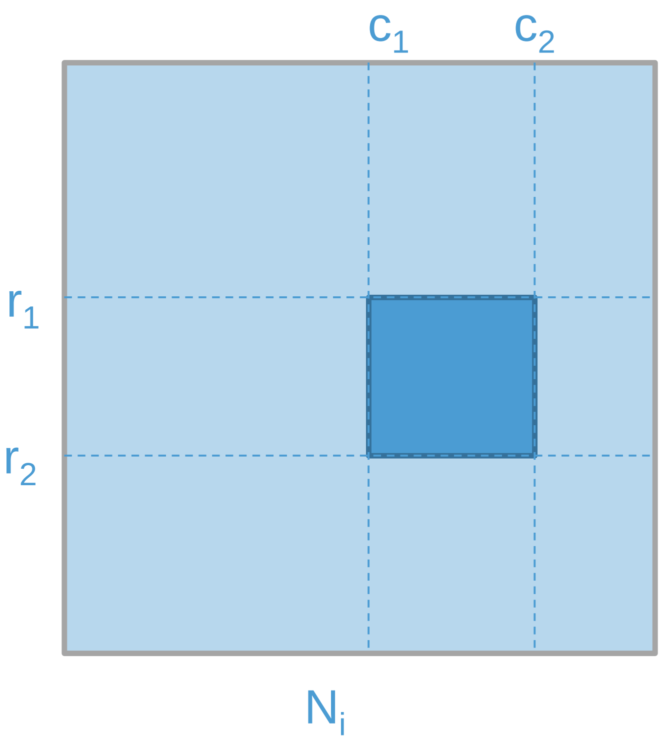


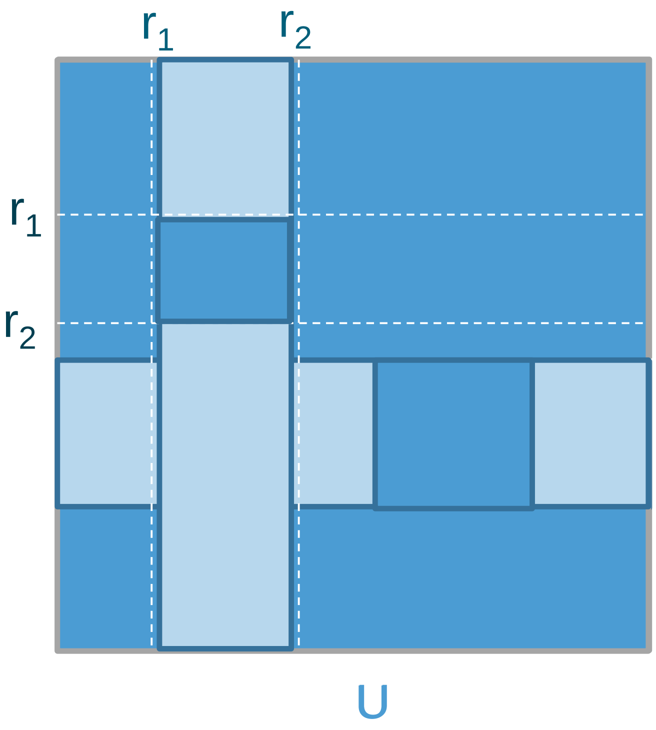


Ensuring The Second Condition

$$\forall i, 0 \leq i \leq k,\ \ \textit{if}\ \mathsf{supp}(N_0) \times \mathsf{supp}(N_i) \leq \mathsf{supp}(N_i),$$
$$\textit{and}\ \mathsf{supp}(N_i) \times \mathsf{supp}(N_0) \leq \mathsf{supp}(N_i).$$

Put 0s in cols r_1 to r_2 except rows r_1 and r_2

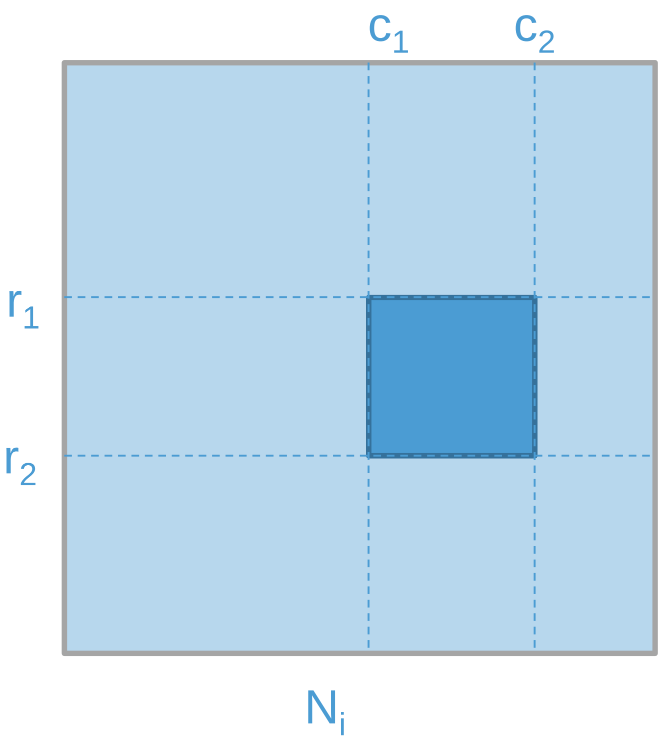


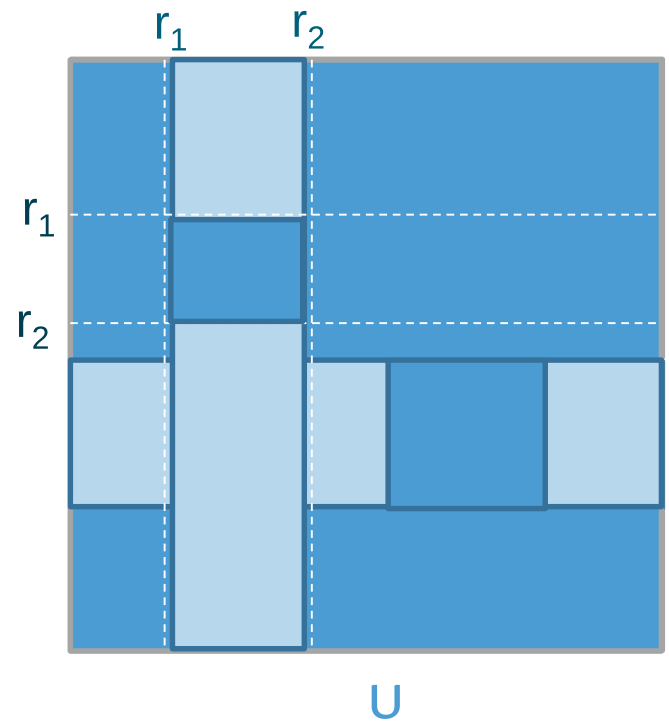


Ensuring The Second Condition

$$\forall i, 0 \leq i \leq k, \ \textit{if} \ \mathsf{supp}(N_0) \times \mathsf{supp}(N_i) \leq \mathsf{supp}(N_i),$$
$$\textit{and} \ \mathsf{supp}(N_i) \times \mathsf{supp}(N_0) \leq \mathsf{supp}(N_i).$$

If $\text{supp}(N_o) \leq U$ and $U \times U \leq U$ then this holds





Introducing Perturbations in the Linear Dynamics

1. $K \leftarrow$ Look for set of all blocks that satisfy conditions (1) and (2)

2. $\Psi \leftarrow$ Largest subset of K that satisfy conditions (1) and (2)

3. $\Lambda \leftarrow$ Induce Faults in Ψ

Robust Reachable Set

Step 1: Given a Matrix

$$\begin{bmatrix} 3 & 2.9 & 3.9 & 2 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Robust Reachable Set

Step 2: Look for largest subset of blocks that satisfy *Cond* (1) and (2)

$$\begin{bmatrix} 3 & \boxed{2.9} & \boxed{3.9} & 2 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Robust Reachable Set

Step 3: User Inputs the amount of variance for the uncertainties

$$\begin{bmatrix} 3 & \boxed{2.9} & \boxed{3.9} & 2 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\pm$10% variance $\quad$ $\pm$20% variance

Robust Reachable Set

Step 4: Introduce uncertain variables

$$\begin{bmatrix} 3 & 2.9x & 3.9y & 2 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Robust Reachable Set

Step 5: Compute Reachable Set of the Uncertain System

$$\begin{bmatrix} 3 & 2.9x & 3.9y & 2 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Illustration

$$\begin{bmatrix} x_1^+ \\ x_2^+ \\ x_3^+ \\ x_4^+ \end{bmatrix} = \begin{bmatrix} 3 & 2.9 & 3.9 & 2 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$\underline{+}$10% variance (2.9) $\underline{+}$20% variance (3.9)

Initial Set:

$$\Theta \triangleq [1,2]\times[2,2]\times[3,3]\times[1,1]$$

Safety Condition:

$$\textit{The system is considered to be safe if at every step the value of } x_1 < 100$$

Illustration

$$\begin{bmatrix} x_1^+ \\ x_2^+ \\ x_3^+ \\ x_4^+ \end{bmatrix} = \begin{bmatrix} 3 & 2.9 & 3.9 & 2 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$\pm$10% variance $\pm$20% variance

$$\underbrace{\begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{N_0} + \underbrace{\begin{bmatrix} 0 & 2.9 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{N_1} y + \underbrace{\begin{bmatrix} 0 & 0 & 3.9 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{N_2} z$$

LME Representation

Illustration

$$\underbrace{\begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{N_0} + \underbrace{\begin{bmatrix} 0 & 2.9 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{N_1} y + \underbrace{\begin{bmatrix} 0 & 0 & 3.9 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{N_2} z$$

$$\forall i, j, 1 \leq i, j \leq k, \operatorname{supp}(N_i) \otimes \operatorname{supp}(N_j) = \boldsymbol{0}$$

$$\forall i, 0 \leq i \leq k, \operatorname{supp}(N_0) \otimes \operatorname{supp}(N_i) \leq \operatorname{supp}(N_i), \\ \mathit{and} \operatorname{supp}(N_i) \otimes \operatorname{supp}(N_0) \leq \operatorname{supp}(N_i).$$

- Check:

Illustration

$$\underbrace{\begin{bmatrix}3 & 0 & 0 & 2\\0 & 7 & 0 & 0\\0 & 0 & 2 & 0\\0 & 0 & 0 & 1\end{bmatrix}}_{N_0} + \underbrace{\begin{bmatrix}0 & 2.9 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}}_{N_1} y + \underbrace{\begin{bmatrix}0 & 0 & 3.9 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}}_{N_2} z$$

$$\forall i, j, 1 \leq i, j \leq k, \mathrm{supp}(N_i) \otimes \mathrm{supp}(N_j) = \boldsymbol{0}$$

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- Check:
  $supp(N_1) \otimes supp(N_1) = 0$

Illustration

$$\underbrace{\begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{N_0} + \underbrace{\begin{bmatrix} 0 & 2.9 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{N_1} y + \underbrace{\begin{bmatrix} 0 & 0 & 3.9 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{N_2} z$$

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- Check:
  $supp(N_1) \otimes supp(N_1) = 0$
  $supp(N_1) \otimes supp(N_2) = 0$

Illustration

$$\underbrace{\begin{bmatrix}3 & 0 & 0 & 2\\ 0 & 7 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}}_{N_0} + \underbrace{\begin{bmatrix}0 & 2.9 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}}_{N_1} y + \underbrace{\begin{bmatrix}0 & 0 & 3.9 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}}_{N_2} z$$

$$\forall i, j, 1 \leq i, j \leq k, \mathrm{supp}(N_i) \otimes \mathrm{supp}(N_j) = \boldsymbol{0}$$

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- Check:

$supp(N_1) \otimes supp(N_1) = 0$
$supp(N_1) \otimes supp(N_2) = 0$
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Illustration

$$\underbrace{\begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{N_0} + \underbrace{\begin{bmatrix} 0 & 2.9 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{N_1} y + \underbrace{\begin{bmatrix} 0 & 0 & 3.9 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{N_2} z$$

$$\forall i, j, 1 \leq i, j \leq k, \text{supp}(N_i) \otimes \text{supp}(N_j) = \boldsymbol{0}$$

$$\forall i, 0 \leq i \leq k, \text{supp}(N_0) \otimes \text{supp}(N_i) \leq \text{supp}(N_i),$$
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- Check:

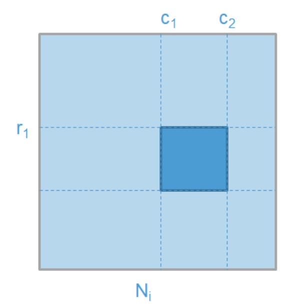
$supp(N_1) \otimes supp(N_1) = 0$
$supp(N_1) \otimes supp(N_2) = 0$
$supp(N_2) \otimes supp(N_2) = 0$
$supp(N_2) \otimes supp(N_1) = 0$

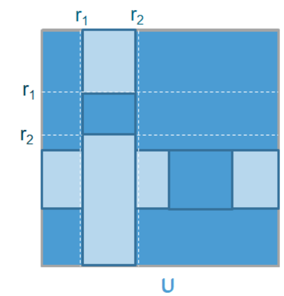
Illustration

$$\underbrace{\begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{N_0} + \underbrace{\begin{bmatrix} 0 & 2.9 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{N_1} y + \underbrace{\begin{bmatrix} 0 & 0 & 3.9 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{N_2} z$$

$$\forall i, j, 1 \leq i, j \leq k, \text{supp}(N_i) \otimes \text{supp}(N_j) = \boldsymbol{0}$$

$$\forall i, 0 \leq i \leq k, \text{supp}(N_0) \otimes \text{supp}(N_i) \leq \text{supp}(N_i),$$
$$\textit{and}\ \text{supp}(N_i) \otimes \text{supp}(N_0) \leq \text{supp}(N_i).$$





- Check:

$$supp(N_1) \otimes supp(N_1) = 0$$
$$supp(N_1) \otimes supp(N_2) = 0$$
$$supp(N_2) \otimes supp(N_2) = 0$$
$$supp(N_2) \otimes supp(N_1) = 0$$

- Compute

$$U_{N_1}$$

Illustration

$$\underbrace{\begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{N_0} + \underbrace{\begin{bmatrix} 0 & 2.9 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{N_1} y + \underbrace{\begin{bmatrix} 0 & 0 & 3.9 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{N_2} z$$

$\forall i, j, 1 \leq i, j \leq k, \text{supp}(N_i) \otimes \text{supp}(N_j) = \boldsymbol{0}$

$\forall i, 0 \leq i \leq k, \text{supp}(N_0) \otimes \text{supp}(N_i) \leq \text{supp}(N_i),$ *and* $\text{supp}(N_i) \otimes \text{supp}(N_0) \leq \text{supp}(N_i).$

- Check:

  $supp(N_1) \otimes supp(N_1) = 0$
  $supp(N_1) \otimes supp(N_2) = 0$
  $supp(N_2) \otimes supp(N_2) = 0$
  $supp(N_2) \otimes supp(N_1) = 0$

- Compute

  U_{N_1}

- Check:

  $supp(N_0) \leq U_{N_1}$

Illustration

$$\underbrace{\begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{N_0} + \underbrace{\begin{bmatrix} 0 & 2.9 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{N_1} y + \underbrace{\begin{bmatrix} 0 & 0 & 3.9 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{N_2} z$$

$$\forall i, j, 1 \le i, j \le k, \mathrm{supp}(N_i) \otimes \mathrm{supp}(N_j) = \boldsymbol{0}$$

$$\forall i, 0 \le i \le k, \mathrm{supp}(N_0) \otimes \mathrm{supp}(N_i) \le \mathrm{supp}(N_i), \textit{ and } \mathrm{supp}(N_i) \otimes \mathrm{supp}(N_0) \le \mathrm{supp}(N_i).$$

- Check:

  $supp(N_1) \otimes supp(N_1) = 0$

  $supp(N_1) \otimes supp(N_2) = 0$

  $supp(N_2) \otimes supp(N_2) = 0$

  $supp(N_2) \otimes supp(N_1) = 0$

- Compute

  U_{N_2}

- Check:

  $supp(N_0) \le U_{N_1}$

  $supp(N_0) \le U_{N_2}$

Illustration

$$\underbrace{\begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{N_0} + \underbrace{\begin{bmatrix} 0 & 2.9 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{N_1} y + \underbrace{\begin{bmatrix} 0 & 0 & 3.9 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{N_2} z$$

$$\forall i, j, 1 \leq i, j \leq k, \mathrm{supp}(N_i) \otimes \mathrm{supp}(N_j) = \boldsymbol{0}$$

$$\forall i, 0 \leq i \leq k, \mathrm{supp}(N_0) \otimes \mathrm{supp}(N_i) \leq \mathrm{supp}(N_i), \quad \textit{and}\ \mathrm{supp}(N_i) \otimes \mathrm{supp}(N_0) \leq \mathrm{supp}(N_i).$$

- Check:

$$supp(N_1) \otimes supp(N_1) = 0$$
$$supp(N_1) \otimes supp(N_2) = 0$$
$$supp(N_2) \otimes supp(N_2) = 0$$
$$supp(N_2) \otimes supp(N_1) = 0$$

- Check:

$$supp(N_0) \leq U_{N_1}$$
$$supp(N_0) \leq U_{N_2}$$
$$U \times U \leq U$$

- Compute

$$U = U_{N_1} \cap U_{N_2}$$

Illustration

$$\underbrace{\begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{N_0} + \underbrace{\begin{bmatrix} 0 & 2.9 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{N_1} y + \underbrace{\begin{bmatrix} 0 & 0 & 3.9 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{N_2} z$$

$$\forall i, j, 1 \leq i, j \leq k, \text{supp}(N_i) \otimes \text{supp}(N_j) = \boldsymbol{0}$$

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- Check:

$supp(N_1) \otimes supp(N_1) = 0$
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$supp(N_2) \otimes supp(N_2) = 0$
$supp(N_2) \otimes supp(N_1) = 0$

- Compute

$$U = U_{N_1} \cap U_{N_2}$$

- Check:

$supp(N_0) \leq U_{N_1}$
$supp(N_0) \leq U_{N_2}$

Can be replaced

$U \times U \leq U$

Illustration

$$\underbrace{\begin{bmatrix}3&0&0&2\\0&7&0&0\\0&0&2&0\\0&0&0&1\end{bmatrix}}_{N_0}+\underbrace{\begin{bmatrix}0&2.9&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}}_{N_1}y+\underbrace{\begin{bmatrix}0&0&3.9&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}}_{N_2}z$$

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$$U \times U \le U$$

Conditions hold!

Illustration

$$\underbrace{\begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{N_0} + \underbrace{\begin{bmatrix} 0 & 2.9 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{N_1} y + \underbrace{\begin{bmatrix} 0 & 0 & 3.9 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{N_2} z$$

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$U \times U \leq U$

Conditions hold!

- Compute

$$U = U_{N_1} \cap U_{N_2}$$

Therefore, from the Theorem we know that for all *m*, A^m is an LME

Illustration

We now compute A^2

$$\begin{bmatrix} 9 & 0 & 0 & 8 \\ 0 & 49 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 29 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} y + \begin{bmatrix} 0 & 0 & 19.5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} z$$

For checking the safety property $x_1 < 100$, we formulate the intersection checking problem as a **bi-linear programming** and use **Gurobi** to solve it

Illustration

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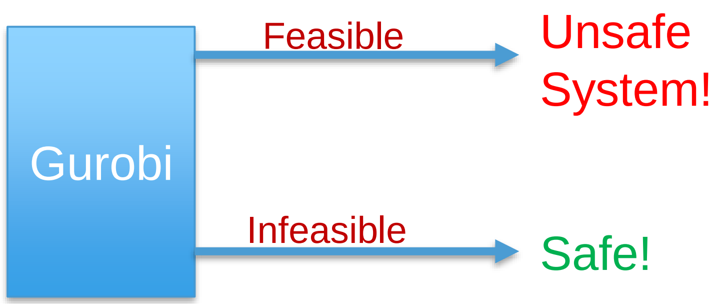
$$\begin{bmatrix} 9 & 0 & 0 & 8 \\ 0 & 49 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 29 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} y + \begin{bmatrix} 0 & 0 & 19.5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} z$$

For checking the safety property $x_1 < 100$, we formulate the intersection checking problem as a **bi-linear programming** and use **Gurobi** to solve it



Summary – Robust Reachable Set

- Take the Matrix as Input

$$\begin{bmatrix} 3 & 2.9 & 3.9 & 2 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Summary – Robust Reachable Set

- Take the Matrix as Input
- Search for the set of blocks satisfying *Cond. (1)* and *(2)*

$$\begin{bmatrix} 3 & \boxed{2.9} & \boxed{3.9} & 2 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Summary – Robust Reachable Set

- Take the Matrix as Input
- Search for the set of blocks satisfying *Cond. (1)* and *(2)*
- Take the Perturbation as Input

$$\begin{bmatrix} 3 & \boxed{2.9} & \boxed{3.9} & 2 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Amount of variance

Summary – Robust Reachable Set

- Take the Matrix as Input
- Search for the set of blocks satisfying *Cond. (1)* and *(2)*
- Take the Perturbation as Input
- Introduce the Uncertain Variables

$$\begin{bmatrix} 3 & 2.9x & 3.9y & 2 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Summary – Robust Reachable Set

- Take the Matrix as Input
- Search for the set of blocks satisfying *Cond. (1)* and *(2)*
- Take the Perturbation as Input
- Introduce the Uncertain Variables
- Perform Reachability Analysis

$$\begin{bmatrix} 3 & 2.9x & 3.9y & 2 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Outline

- Motivation
- Problems due uncertainties in *verification*
- Sufficient conditions to ensure limited effect of uncertainties in the system
- Introduction of *uncertainties* in a system
- **Evaluation**

Evaluation - Anesthesia

$$\frac{1}{V_1} = 8.72 \times 10^{-7}$$

$$\begin{bmatrix} \dot{c_p} \\ \dot{c_1} \\ \dot{c_2} \\ \dot{c_e} \\ \dot{u} \end{bmatrix} = \begin{pmatrix} -(k_{10} + k_{12} + k_{13}) & k_{12} & k_{13} & 0 & 1/V_1 \\ k_{21} & -k_{21} & 0 & 0 & 0 \\ k_{31} & 0 & -k_{31} & 0 & 0 \\ k_d & 0 & 0 & -k_d & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} c_p \\ c_1 \\ c_2 \\ c_e \\ u \end{bmatrix}$$

Evaluation - Anesthesia

$$\begin{bmatrix} \dot{c_p} \\ \dot{c_1} \\ \dot{c_2} \\ \dot{c_e} \\ \dot{u} \end{bmatrix} = \begin{pmatrix} -(k_{10}+k_{12}+k_{13}) & k_{12} & k_{13} & 0 & 1/V_1 \\ k_{21} & -k_{21} & 0 & 0 & 0 \\ k_{31} & 0 & -k_{31} & 0 & 0 \\ k_d & 0 & 0 & -k_d & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} c_p \\ c_1 \\ c_2 \\ c_e \\ u \end{bmatrix}$$

$$\frac{1}{V_1} = 8.72 \times 10^{-7}$$

Condition

$At\ every\ step\ the\ value\ of\ c_p \leq 0$

Analysis

Initial Set

$$\Theta = [1,6] \times [0,10] \times [0,10] \times [1,8] \times [1,1]$$

Evaluation - Anesthesia

$$\begin{bmatrix}\dot{c_p}\\ \dot{c_1}\\ \dot{c_2}\\ \dot{c_e}\\ \dot{u}\end{bmatrix} = \begin{pmatrix} -(k_{10}+k_{12}+k_{13}) & k_{12} & k_{13} & 0 & 1/V_1 \\ k_{21} & -k_{21} & 0 & 0 & 0 \\ k_{31} & 0 & -k_{31} & 0 & 0 \\ k_d & 0 & 0 & -k_d & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} c_p \\ c_1 \\ c_2 \\ c_e \\ u \end{bmatrix}$$

$$\frac{1}{V_1} = 8.72 \times 10^{-7}$$

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Analysis

Initial Set

$\Theta = [1,6] \times [0,10] \times [0,10] \times [1,8] \times [1,1]$

No violation till 2000 steps!

Time taken: 3.48 s

Evaluation - Anesthesia

$$\begin{bmatrix} \dot{c_p} \\ \dot{c_1} \\ \dot{c_2} \\ \dot{c_e} \\ \dot{u} \end{bmatrix} = \begin{pmatrix} -(k_{10}+k_{12}+k_{13}) & k_{12} & k_{13} & 0 & 1/V_1 \\ k_{21} & -k_{21} & 0 & 0 & 0 \\ k_{31} & 0 & -k_{31} & 0 & 0 \\ k_d & 0 & 0 & -k_d & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} c_p \\ c_1 \\ c_2 \\ c_e \\ u \end{bmatrix}$$

$$\frac{1}{V_1} = 8.72 \times 10^{-7}$$

$$\frac{1}{V_1} \in [-8.72 \times 10^{-4}, 2 \times 8.72 \times 10^{-4}]$$

Introduce Perturbation!

Evaluation - Anesthesia

$$\begin{bmatrix} \dot{c_p} \\ \dot{c_1} \\ \dot{c_2} \\ \dot{c_e} \\ \dot{u} \end{bmatrix} = \begin{pmatrix} -(k_{10}+k_{12}+k_{13}) & k_{12} & k_{13} & 0 & 1/V_1 \\ k_{21} & -k_{21} & 0 & 0 & 0 \\ k_{31} & 0 & -k_{31} & 0 & 0 \\ k_d & 0 & 0 & -k_d & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} c_p \\ c_1 \\ c_2 \\ c_e \\ u \end{bmatrix}$$

$$\frac{1}{V_1} = 8.72 \times 10^{-7}$$

$$\frac{1}{V_1} \in [-8.72 \times 10^{-4}, 2 \times 8.72 \times 10^{-4}]$$

Condition

At every step the value of $c_p \leq 0$ ***(Same as before)***

Analysis

Initial Set

$\Theta = [1,6] \times [0,10] \times [0,10] \times [1,8] \times [1,1]$ ***(Same as before)***

Violation at 623rd step!

Time taken: 2.78 s

Evaluation

Red: Violation
Green: No-Violation (up-to 2000 steps)
w/o: Without Uncertainties

Benchmark	Dim	Fault Search Time	#Uncertainties	Time
Quadcopter (w/o)	16			0.69s
Quadcopter	16	8.87s	60	1.35s
Platoon (w/o)	10			7.8s
Platoon	10	0.49s	9	1.76s
Anesthesia (w/o)	5			3.48s
Anesthesia	5	0.01s	4	2.78s
Motor (w/o)	7			5.04s
Motor	7	0.01s	12	0.08s

Evaluation

Dimension of the matrix A is n × n

Benchmark	Dim	Fault Search Time	#Uncertainties	Time
Quadcopter (w/o)	16			0.69s
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Anesthesia	5	0.01s	4	2.78s
Motor (w/o)	7			5.04s
Motor	7	0.01s	12	0.08s

Evaluation

Time taken to search for all possible places where uncertainty can be introduced in the matrix A

Benchmark	Dim	Fault Search Time	#Uncertainties	Time
Quadcopter (w/o)	16			0.69s
Quadcopter	16	8.87s	60	1.35s
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Evaluation

Number of uncertainties introduced

Benchmark	Dim	Fault Search Time	#Uncertainties	Time
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Evaluation

Time taken to perform the safety check

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Evaluation

Thank You!

bineet@cs.unc.edu

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